Math 226 **(White)** **Final Exam**
Integration Techniques and Applications **October 13, 2017**

Do NOT turn over this page until instructed to begin.

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Section: ∞

Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Solutions presented without sufficient supporting work may receive no credit. Your work must be legible, and numerical answers should be presented as exact mathematical expressions, simplified as appropriate, and not by a decimal approximation, unless explicitly required by the problem.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. You may not use methods that we have not covered.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you MUST stop working when your proctor tells you to stop.

1	2	3	4	5	6	7	Total
32 pts	21 pts	5 pts	6 pts	21 pts	5 pts	10 pts	100 pts

1. (32 points) Compute the following integrals. You must show correct work to receive full credit. Explicitly show your substitutions and be careful not to omit the absolute value for logarithm integrals. If needed, you may use the facts that $\int \sec(x)\,dx = \ln(|\sec(x)+\tan(x)|)+C$ and $\int \sec^3(x) = \frac{1}{2}\sec(x)\tan(x)+\frac{1}{2}\ln(|\sec(x)+\tan(x)|)+C$.

(a) $\displaystyle\int \frac{3x-1}{x^2-x-6}\,dx$

$$\frac{3x-1}{x^2-x-6} = \frac{3x-1}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$
$$= \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}$$

$A+B=3$
$2A-3B=-1$
$3A+3B=9$
$5A=8$
$A=8/5$

$2A-3B=-1$
$2A+2B=6$
$-5B=-7$
$B=7/5$

$$= \int \left(\frac{8}{5}\right)\frac{1}{x-3} + \left(\frac{7}{5}\right)\frac{1}{x+2}\,dx$$
$$= \frac{8}{5}\ln(|x-3|)+\frac{7}{5}\ln(|x+2|)+C$$

(b) $\displaystyle\int \frac{2x^3-7x^2+9x-2}{x^2-2x+1}\,dx$

$$\begin{array}{r|l} & 2x-3 \\ \hline x^2-2x+1 & 2x^3-7x^2+9x-2 \\ & 2x^3-4x^2+2x \\ \hline & -3x^2+7x-2 \\ & -3x^2+6x-3 \\ \hline & x+1 \end{array}$$

$$= \int 2x-3+\frac{x+1}{x^2-2x+1}\,dx$$
$$= x^2-3x+\frac{1}{2}\int\frac{2x+2}{x^2-2x+1}\,dx$$
$$= x^2-3x+\frac{1}{2}\int\frac{2x-2}{x^2-2x+1}\,dx+\frac{1}{2}\int\frac{4}{x^2-2x+1}\,dx$$
$$= x^2-3x+\frac{1}{2}\ln(|x^2-2x+1|)+2\int\frac{1}{(x-1)^2}\,dx$$
$$= x^2-3x+\frac{1}{2}\ln(|x^2-2x+1|)+2\left(\frac{-1}{x-1}\right)+C$$

(c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\, dx$

$$= \int \frac{5\sec(\theta(x))\tan(\theta(x))}{25\sec^2(\theta(x))\,5\tan(\theta(x))}\,\theta'(x)\,dx$$

$$= \frac{1}{25}\int \frac{1}{\sec(\theta)}\, d\theta$$

$$= \frac{1}{25}\int \cos(\theta)\, d\theta$$

$$= \frac{1}{25}\sin(\theta(x)) + C$$

$$= \frac{1}{25}\,\frac{\sqrt{x^2-25}}{x} + C$$

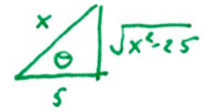
Trig ID

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

$$25\sec^2(\theta) - 25 = 25\tan^2(\theta)$$

Define $\theta(x)$ implicitly:

$$x = 5\sec(\theta(x))$$



$$x^2 - 25 = 25\sec^2(\theta(x)) - 25$$

$$= 25\tan^2(\theta(x))$$

Differentiate:

$$1 = 5\sec(\theta(x))\tan(\theta(x))\,\theta'(x)$$

(or)

$$dx = 5\sec(\theta)\tan(\theta)\,d\theta$$

$$x^2 = 25\sec^2(\theta(x))$$

(d) $\displaystyle\int_0^3 \frac{1}{\sqrt{x^2+9}}\, dx$

$$= \int_0^3 \frac{3\sec^2(\theta(x))}{3\sec(\theta(x))}\,\theta'(x)\,dx$$

$$= \int_0^3 \sec(\theta(x))\,\theta'(x)\,dx$$

$$= \int_{\theta(0)}^{\theta(3)} \sec(\theta)\,d\theta$$

$$= \int_0^{\pi/4} \sec(\theta)\,d\theta$$

$$= \ln\left(|\sec(\theta) + \tan(\theta)|\right)\Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0)$$

$$= \ln(\sqrt{2}+1)$$

Trig ID:

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$9\tan^2(\theta) + 9 = 9\sec^2(\theta)$$

Define $\theta(x)$ implicitly:

$$x = 3\tan(\theta(x))$$

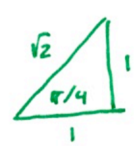
$$1 = 3\sec^2(\theta(x))\,\theta'(x)$$

(or)

$$dx = 3\sec^2(\theta)\,d\theta$$

$$x^2 + 9 = 9\tan^2(\theta(x)) + 9$$

$$= 9\sec^2(\theta(x))$$



$$0 = 3\tan(\theta(0)) \quad \text{so } \theta(0) = 0$$

$$3 = 3\tan(\theta(3))$$

$$1 = \tan(\theta(3)) \quad \text{so } \theta(3) = \pi/4$$

2. (21 points) If possible, compute the following integrals. If the integral is *improper*, be sure to show the appropriate limits and indicate clearly why the integral converges or diverges.

(a) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln(x))^2}\, dx$

$$\lim_{t\to\infty} \int_e^t \frac{1}{x(\ln(x))^2}\, dx$$

$$= \lim_{t\to\infty} \left(\frac{-1}{\ln(x)}\right)\Big|_e^t$$

$$= \lim_{t\to\infty} \frac{-1}{\ln(t)} + \frac{1}{\ln(e)} = 0 + 1$$

$$= 1$$

$u(x) = \ln(x)$

$u'(x) = \frac{1}{x}$

$$\int \frac{1}{x(\ln(x))^2}\, dx$$

$$= \int \frac{1}{u^2(x)} \cdot u'(x)\, dx$$

$$= \int \frac{1}{u^2}\, du$$

$$= -\frac{1}{u(x)} + C$$

$$= \frac{-1}{\ln(x)} + C$$

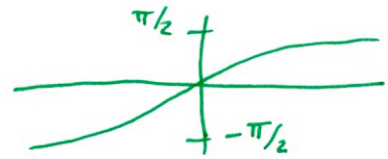
(b) $\displaystyle\int_{-\infty}^{\sqrt{3}} \frac{1}{x^2+1}\, dx$

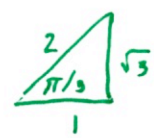
$$\lim_{t\to-\infty} \int_t^{\sqrt{3}} \frac{1}{x^2+1}\, dx$$

$$= \lim_{t\to-\infty} \arctan(x)\Big|_t^{\sqrt{3}}$$

$$= \lim_{t\to-\infty} \arctan(\sqrt{3}) - \arctan(t) = \frac{\pi}{3} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$$

$\arctan(t)$





* (c) $\displaystyle\int_0^{3} \frac{1}{(x-1)^{4/3}}\, dx$

Improper because there is a discontinuity at $x=1$.

Check integrals separately.

$$\int_1^3 \frac{1}{(x-1)^{4/3}}\, dx = \lim_{t\to1^+} \int_t^3 \frac{1}{(x-1)^{4/3}}\, dx = \lim_{t\to1^+} 3(x-1)^{-1/3}\Big]_t^3$$

$$= \lim_{t\to1^+} \frac{3}{\sqrt[3]{2}} - \frac{3}{\sqrt[3]{t-1}} = -\infty$$

(*)

Integral does not converge.

3. (5 points) **Circle** the correct form for the partial fraction decomposition of the rational function

$$\frac{x^7+5x^4-13x^2+x-2}{(x^2+x+2)^2(x^2+x-2)^2(x+4)^3}$$

Note:
$(x^2+x-2)^2 = (x-1)^2(x+2)^2$
(x^2+x+2) not factorable

(a) $\dfrac{Ax+B}{x^2+x+2}+\dfrac{Cx+D}{(x^2+x+2)^2}+\dfrac{Ex^2+Fx+G}{(x+4)^3}$

(b) $\dfrac{Ax+B}{x^2+x+2}+\dfrac{Cx+D}{x^2+x+2}+\dfrac{E}{(x+4)}+\dfrac{F}{(x+4)^2}+\dfrac{G}{(x+4)^3}$

(c) $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2}+\dfrac{Ex+F}{x^2+x+2}+\dfrac{Gx+H}{(x^2+x+2)^2}+\dfrac{J}{x+4}+\dfrac{K}{(x+4)^2}+\dfrac{L}{(x+4)^3}$

(d) $\dfrac{Ax+B}{x^2+x+2}+\dfrac{Cx+D}{(x^2+x+2)^2}+\dfrac{Ex+F}{x^2+x-2}+\dfrac{Gx+H}{(x^2+x-2)^2}+\dfrac{J}{(x+4)}+\dfrac{K}{(x+4)^2}+\dfrac{L}{(x+4)^3}$

(e) $\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{Cx+D}{x^2+x+2}+\dfrac{Ex+F}{(x^2+x+2)^2}+\dfrac{G}{x+4}$

(f) $\dfrac{Ax+B}{(x^2+x+2)^2}+\dfrac{Cx+D}{(x^2+x-2)^2}+\dfrac{E}{(x+4)}+\dfrac{F}{(x+4)^2}+\dfrac{G}{(x+4)^3}$

4. (6 points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. **Do NOT compute the exact value of the integral**, but **DO** show all the work needed to apply the Comparison Theorem, in particular, the *inequality* used.

$$\int_1^\infty \frac{1}{\sqrt[3]{x^7+4x^5+9}}\,dx$$

Observe:

$$0 \le \frac{1}{\sqrt[3]{x^7+4x^5+9}} \le \frac{1}{\sqrt[3]{x^7}}$$

Since $\int_1^\infty \frac{1}{\sqrt[3]{x^7}}\,dx = \int_1^\infty \frac{1}{x^{7/3}}\,dx$ converges,

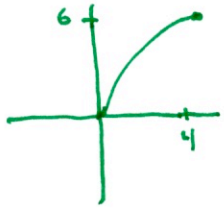
as it is of the form $\int_1^\infty \frac{1}{x^p}\,dx$ with $p = 7/3 > 1$,

by the comparison theorem

$\int_1^\infty \frac{1}{\sqrt[3]{x^7+4x^5+9}}\,dx$ also converges.

5. (21 points) For both parts of this question, let C be the curve given by $y = 3\sqrt{x}$ for $0 \le x \le 4$.

(a) Set up the integral for the arc length of the curve C.
Do NOT try to evaluate or simplify the integral.



$$\int_0^4 \sqrt{1+\left(\frac{3}{2\sqrt{x}}\right)^2}\,dx = \int_0^4 \sqrt{1+\frac{9}{4x}}\,dx$$

$f(x) = 3\sqrt{x}$

$f'(x) = \frac{3}{2\sqrt{x}}$

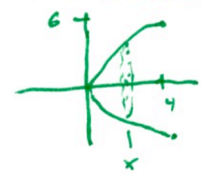
$\underline{\text{or}}$

$$\int_0^6 \sqrt{1+\left(\frac{2y}{9}\right)^2}\,dy = \int_0^6 \sqrt{1+\frac{4y^2}{81}}\,dy$$

$g(y) = \left(\frac{y}{3}\right)^2$

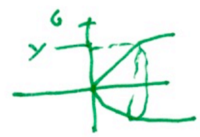
$g'(y) = \frac{2y}{9}$

(b) Set up the integral for the area of the **surface** obtained by rotating the curve C about the x-axis. **Do NOT try to evaluate or simplify the integral.**



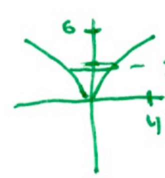
$$\int_0^4 2\pi(3\sqrt{x})\sqrt{1+\frac{9}{4x}}\,dx$$

$\underline{\text{or}}$



$$\int_0^6 2\pi y\sqrt{1+\frac{4y^2}{81}}\,dy$$

(c) Set up the integral for the area of the **surface** obtained by rotating the curve C about the y-axis. **Do NOT try to evaluate or simplify the integral.**



$$\int_0^6 2\pi\left(\frac{y}{3}\right)^2\sqrt{1+\frac{4y^2}{81}}\,dy$$

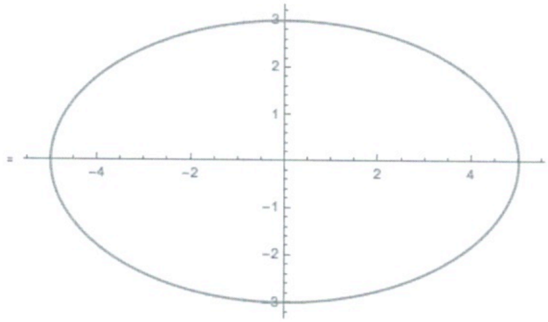
or

$$\int_0^4 2\pi x\sqrt{1+\frac{9}{4x}}\,dx$$

6. (5 points) As pictured below, the graph of the equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

is an ellipse centered at the origin with horizontal axis of length 10 and vertical axis of length 6. Express the **top half** of this ellipse as a parametric curve with parameter t. In the illustration, indicate the initial point and final point and direction of the curve using your parametrization.



① $\frac{y^2}{9} = 1 - \frac{x^2}{25}$

$y^2 = 9\left(1 - \frac{x^2}{25}\right)$

$y = 3\sqrt{1 - \frac{x^2}{25}}$ because $y \geq 0$

$x = f(t) = t$

$y = g(t) = 3\sqrt{1 - \frac{t^2}{25}}$ for $t \in [-5, 5]$

Initial point $(-5, 0)$ Final point $(5, 0)$

② Use $\sin^2(t) + \cos^2(t) = 1$

$\frac{25\cos^2(t)}{25} + \frac{9\sin^2(t)}{9} = 1$

$f(t) = x = 5\cos(t)$

$g(t) = y = 3\sin(t)$ for $t \in [0, \pi]$

Initial point: $(5, 0)$

Final point: $(-5, 0)$

7. (10 points) For this entire problem, the curve C is given parametrically by the functions below for $0 \le t \le \pi/2$:

$$x = f(t) = 4\sin(t)$$
$$y = g(t) = \cos(2t).$$

(a) Find the slope of the tangent to C when $t = \pi/4$.

$$f'(t) = 4\cos(t)$$
$$g'(t) = -2\sin(2t)$$
$$m(t) = \frac{g'(t)}{f'(t)} = \frac{-2\sin(2t)}{4\cos(t)} = \frac{-\sin(2t)}{2\cos(t)}$$
$$m\left(\frac{\pi}{4}\right) = \frac{-\sin(\pi/2)}{2\cos(\pi/4)} = \frac{-1}{2(1/\sqrt{2})}$$
$$= -\frac{\sqrt{2}}{2}$$
$$= \frac{-1}{\sqrt{2}}$$

(b) Set up the integral with respect to the parameter t for the arc length of the curve C.
Do NOT try to evaluate or simplify the integral.

$$\int_0^{\pi/2} \sqrt{(4\cos(t))^2 + (-2\sin(2t))^2}\, dt$$
$$= \int_0^{\pi/2} \sqrt{16\cos^2(t) + 4\sin^2(2t)}\, dt$$