Math 323 Midterm Examination 2, Sample 1

Problem 1. Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+4y^2}$$

Problem 2. a) Find the linear approximation near the point $(1, 2, 3)$ to the function $F(x, y, z) = x^2 + yz$.

b) Find the directional derivative of the function F at the point $(1, 2, 3)$ in the direction of the point $(3, 4, 4)$.

Problem 3. Find and classify all critical points of the function

$$F(x, y) = x^2 - xy + y + 3$$

Problem 4. Evaluate the iterated integral.

$$\int_{-1}^{1} \int_{1}^{2} y \cdot \ln(x + y^2)\ dx\ dy$$

Problem 5. Function $z(x, y)$ is given implicitly by the equation

$$xy + z^3 = xz + y$$

a) Find $\frac{\partial z}{\partial x}$ at the point $(x, y, z) = (1, 2, -1)$.

b) Find $\frac{\partial z}{\partial y}$ at the point $(x, y, z) = (1, 2, -1)$.

c) Find the equation of the tangent plane to the graph of $z(x, y)$ at the point $(x, y, z) = (1, 2, -1)$.

Problem 6. Evaluate the double integral

$$\iint\limits_{D} \frac{x^2}{x^2+y^2+1}\, dA$$

where D is given by the inequality $x^2+y^2 \le 4$

Problem 7. Find the absolute maximum and absolute minimum values of the function

$$F(x, y) = x^2 - y^2$$

in the region $D = \{(x, y)|x^2 + y^2 - 2y \leq 3\}$

Math 323 Midterm Examination 2, Sample 2

Problem 1. Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

(**Hint:** rewrite in polar coordinates).

Problem 2. a) Find the equation of the tangent plane at the point above $(1,2)$ to the graph of the function $F(x,y) = x^2y + y^3$.

b) Find the directional derivative of the function F at the point $(1,2)$ in the direction of the point $(5,-1)$.

Problem 3. Evaluate the double integral

$$\iint\limits_D \frac{x}{x^2+y^2+1}\,dA$$

where D is given by the inequalities $x^2+y^2 \leq 4$, $x \geq 0$.

Problem 4. Evaluate the iterated integral.

$$\int\limits_0^2 \int\limits_y^2 \frac{1}{\sqrt{x^2+5}}\,dx\;dy$$

Problem 5. Find all points on the surface

$$2xy + z^2 = 10$$

where the tangent plane is parallel to the plane $3x + y - 2z = 5$.

Problem 6. a) Find and classify all critical points of the function

$$F(x,y) = x^2 + y^2(1-x)^3$$

b) Find the limit (possibly infinite)

$$\lim_{x\to+\infty} F(x,0)$$

b) Find the limit (possibly infinite)

$$\lim_{x\to+\infty} F(x,1)$$

Problem 7. Find and **simplify** F_{xy} for

$$F(x,y,z) = \frac{x-y}{xy - xz - yz + z^2}$$

Answer the questions in the spaces provided on the question sheets. Explain your answers and reasoning

Name: ______________________________

Section: ______________________________

Instructions:

- Read problems very carefully. If you have any questions raise your hand.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. In this way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.
- You must use a non-erasable pen. Do not use a pencil!
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand
- There is no need to reduce fractions like $\frac{3}{6}$ to lowest terms.
- Trigonometric functions should be simplified when possible.
- Circle your final answer.

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	15	15	15	20	15	100
Score:								

1. (10 points) Find the limit, or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

2. (10 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $yz + x\ln y = z^2$

3. (15 points) Let $f(x, y) = xe^{xy}$, Find:

   a) the value of f_x and f_y at $P = (1, 0)$

   b) the tangent plane of $z = f(x, y)$ passes $Q = (1, 0, 1)$

4. (15 points) For function $f(x, y, z) = x^2y + y^2z$ at $P = (1, 2, 3)$

a) Find the directional derivative of f at P in direction $\mu =< 2, -1, 2 >$.

b) Find the maximum rate of change of f at P and the direction in which it occurs.

5. (15 points) Let $w = x^3 y + y^2$, where $x = rse^t$ and $y = rs^2 \cos(t)$. Using the chain rule to find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ at $r = 1, s = 2, t = 0$

6. (20 points) For function $f(x, y) = x^3 - 3x + 3xy^2$,

a) Find the linearization of f at (1, 1).

b) Find and classify (as local maximum, local minimum or saddle point) all critical points of f.

7. (15 points) Find the absolute maximum and minimum of $f(x,y) = x^2 - y^2 + y$ on the unit disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

Math 323 Midterm Examination 2, Sample 1

Problem 1. Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+4y^2}$$

$$F(x,y) = \frac{x^2}{x^2+4y^2}$$

$$\lim_{x\to 0} F(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

$$\neq \quad \boxed{\text{So the limit does not exist}}$$

$$\lim_{y\to 0} F(0,y) = \lim_{y\to 0} \frac{0}{4y^2} = 0$$

Problem 2. a) Find the linear approximation near the point $(1, 2, 3)$ to the function $F(x, y, z) = x^2 + yz$.

b) Find the directional derivative of the function F at the point $(1, 2, 3)$ in the direction of the point $(3, 4, 4)$.

$F(1,2,3) = 7$

a) $\nabla F = \langle 2x, z, y \rangle \overset{(1,2,3)}{=} \langle 2, 3, 2 \rangle, \quad \boxed{F \approx 7 + 2(x-1) + 3(y-2) + 2(z-2)}$

b) $$\vec{u} = \frac{\langle 3-1, 4-2, 4-3 \rangle}{|\langle 3-1, 4-2, 4-3 \rangle|} = \frac{\langle 2, 2, 1 \rangle}{3} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

$$F_{\vec{u}} = \nabla F \cdot \vec{u} = \frac{4}{3} + 2 + \frac{2}{3} = \boxed{4}$$

Problem 3. Find and classify all critical points of the function

$$F(x, y) = x^2 - xy + y + 3$$

$$\nabla F = \langle 2x-y, -x+1 \rangle$$

$$\begin{cases} 2x-y=0 \\ -x+1=0 \end{cases} \quad \text{So } x=1,\ y=2$$

$$F_{xx} = 2, \quad F_{xy} = F_{yx} = -1, \quad F_{yy} = 0$$

$$\det \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0 \quad \text{So} \quad \boxed{(1,2) \text{ is a saddle}}$$

Problem 4. Evaluate the iterated integral.

$$\int_{-1}^{1}\int_{1}^{2} y \cdot \ln(x + y^2)\, dx\, dy$$

By Fubini's Theorem, this equals

$$\int_1^2 \int_{-1}^{1} y \ln(x+y^2)\, dy\, dx = \int_1^2 0\, dx = \boxed{0}$$

$$\int_{-1}^{1} y \ln(x+y^2)\, dy = \int_{x+1}^{x+1} \frac{1}{2} \ln(u)\, du = 0$$

$u = x+y^2$

$du = 2y\, dy$

Problem 5. Function $z(x, y)$ is given implicitly by the equation

$$xy + z^3 = xz + y$$

a) Find $\frac{\partial z}{\partial x}$ at the point $(x, y, z) = (1, 2, -1)$.

<u>y is fixed</u>

$$y + 3z^2\frac{\partial z}{\partial x} = z + x\cdot\frac{\partial z}{\partial x}$$

$$2 + 3\frac{\partial z}{\partial x} = -1 + \frac{\partial z}{\partial x}$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{3}{2}}$$

b) Find $\frac{\partial z}{\partial y}$ at the point $(x, y, z) = (1, 2, -1)$.

<u>x is fixed</u>

$$x + 3z^2\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial y} + 1$$

$$1 + 3\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} + 1$$

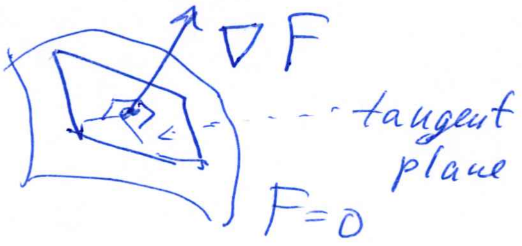
$$\boxed{\frac{\partial z}{\partial y} = 0}$$

c) Find the equation of the tangent plane to the graph of $z(x, y)$ at the point $(x, y, z) = (1, 2, -1)$.

We can use a) and b) or do the following:

$$F(x,y,z) = xy + z^3 - xz - y.$$

$$\nabla F = \langle y-z, x-1, 3z^2-x \rangle$$



$$\nabla F = \langle 3, 0, 2 \rangle$$

So: $$3(x-1) + 0(y-2) + 2(z+1) = 0$$

$$\boxed{3x + 2z - 1 = 0}$$

$$\left(\text{Or } 3x + 2z = 1 \right)$$

Problem 6. Evaluate the double integral

$$\iint_D \frac{x^2}{x^2+y^2+1}\,dA$$

where D is given by the inequality $x^2+y^2 \leq 4$

$$\iint_D \frac{x^2}{x^2+y^2+1}\,dA = \int_0^2\int_0^{2\pi} \frac{r^2\cos^2\theta}{r^2+1}\cdot r\,d\theta\,dr$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_0^{2\pi} \frac{r^3}{r^2+1}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{r^3}{2(r^2+1)}(1+\cos 2\theta)\,d\theta =$$

$$= \frac{r^3}{2(r^2+1)}\cdot\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = \frac{r^3\pi}{r^2+1}$$

So: $$\int_0^2 \frac{r^3\cdot\pi}{r^2+1}\,dr = \pi\int_0^2\left(r-\frac{r}{r^2+1}\right)dr =$$

$$= \pi\left(\frac{r^2}{2}-\frac{1}{2}\ln(r^2+1)\right)\Big|_0^2$$

$$= \pi\left(2-\frac{1}{2}\ln 5-0\right)$$

$$= \boxed{\frac{4-\ln 5}{2}\pi}$$

$$\begin{array}{r|l} & r \;\; R(-r) \\ r^2+1 & r^3 \\ & -(r^3+r) \\ \hline & -r \end{array}$$

Problem 7. Find the absolute maximum and absolute minimum values of the function

$$F(x, y) = x^2 - y^2$$

in the region $D = \{(x, y)|x^2 + y^2 - 2y \leq 3\}$

Inside D

$\nabla F = \langle 2x, -2y \rangle$

$\nabla F = \vec{0}$ $\quad \begin{cases} 2x = 0 \\ -2y = 0 \end{cases}$ $\quad (0,0)$. It is in D.

Boundary:

$G(x,y) = x^2 + y^2 - 2y - 3 = 0$

$\nabla G = \langle 2x, 2y-2 \rangle \qquad \nabla F = \lambda \nabla G$

$$\begin{cases} 2x = \lambda \cdot 2x \quad \longrightarrow \boxed{x=0} \text{ or } \boxed{\lambda = 1} \\ -2y = \lambda(2y-2) \\ x^2 + y^2 - 2y = 3 \end{cases}$$

If $x=0$, then: $0^2 + y^2 - 2y = 3$, so $y^2 - 2y - 3 = 0$, $y = -1, 3$

$(0,-1)$, $(0,3)$

If $\lambda = 1$: $-2y = 1 \cdot (2y-2)$, $y = \frac{1}{2}$

So $x^2 + \frac{1}{4} - 1 = 3$, $x^2 = \frac{15}{4}$, $x = \pm \frac{\sqrt{15}}{2}$ $\left(-\frac{\sqrt{15}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{15}}{2}, \frac{1}{2}\right)$

Finally, $F(0,0) = 0$

absolute minimum

$F(0,-1) = -1$

$F(0,3) = \boxed{-9} \longleftarrow$

$F\left(-\frac{\sqrt{15}}{2}, \frac{1}{2}\right) = \frac{15}{4} - \frac{1}{4} = \frac{7}{2} \longleftarrow$ absolute maximum

$F\left(\frac{\sqrt{15}}{2}, \frac{1}{2}\right) = \frac{7}{2} \longleftarrow$

Math 323 Midterm Examination 2, Sample 2

Problem 1. Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \quad = \lim_{r\to 0}\left(r\cos^3\theta\right)$$

(**Hint:** rewrite in polar coordinates).

$$\frac{x^3}{x^2+y^2} = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta$$

$$\begin{tikzcd}
-r \le & r\cos^3\theta \arrow[d, squiggly] & \le r \\
& 0 \arrow[from=1-1, dr] \arrow[from=1-3, dl] &
\end{tikzcd}$$

So by Squeeze Theorem

$$\lim_{r\to 0}\left(r\cos^3\theta\right) = \boxed{0}$$

Problem 2. a) Find the equation of the tangent plane at the point above $(1,2)$ to the graph of the function $F(x,y) = x^2y + y^3$.

$F(1,2) = 10$

$\nabla F = \langle 2xy, x^2+3y^2\rangle$

$\nabla F(1,2) = \langle 4, 13\rangle$

So: $z = 10 + 4(x-1) + 13(y-2)$

Or: $\boxed{4x + 13y - z = 20}$

b) Find the directional derivative of the function F at the point $(1,2)$ in the direction of the point $(5,-1)$.

$\overrightarrow{(1,2)(5,-1)} = \langle 4,-3\rangle$

$\vec{u} = \frac{1}{5}\langle 4,-3\rangle = \langle \frac{4}{5}, -\frac{3}{5}\rangle$

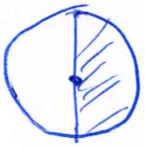
$\nabla F(1,2) = \langle 4,13\rangle$

So $F_{\vec{u}} = \langle 4,13\rangle \cdot \langle \frac{4}{5}, \frac{-3}{5}\rangle = \frac{16}{5} - \frac{39}{5} = \boxed{-\frac{23}{5}}$

Problem 3. Evaluate the double integral

$$\iint_D \frac{x}{x^2+y^2+1}\, dA$$

where D is given by the inequalities $x^2+y^2 \le 4$, $x \ge 0$.



$0 \le r \le 2$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

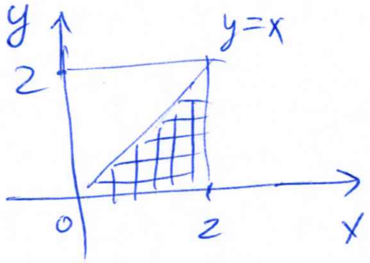
$$\int_{-\pi/2}^{\pi/2}\int_0^2 \frac{r\cos\theta}{r^2+1} \cdot r\,dr\,d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta \cdot \int_0^2 \frac{r^2}{r^2+1}\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta \cdot \int_0^2 \left(1-\frac{1}{r^2+1}\right) dr\, d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta \left(r-\arctan r\right)\Big|_0^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta \left(2-\arctan 2\right) d\theta = \left(2-\arctan 2\right)\cdot \sin\theta \Big|_{-\pi/2}^{\pi/2} = \boxed{4-2\arctan 2}$$

Problem 4. Evaluate the iterated integral.

$$\int_0^2 \int_y^2 \frac{1}{\sqrt{x^2+5}}\, dx\, dy$$



By Fubini Theorem:

$$\int_0^2 \int_0^x \frac{1}{\sqrt{x^2+5}}\, dy\, dx = \int_0^2 \left(\frac{1}{\sqrt{x^2+5}} \cdot y\Big|_0^x\right) dx = \int_0^2 \frac{x}{\sqrt{x^2+5}}\, dx =$$

$u = x^2+5$

$du = 2x\,dx$

$$= \int_5^9 \frac{\frac{1}{2}\,du}{\sqrt{u}} = \sqrt{u}\Big|_5^9 = \boxed{3-\sqrt{5}}$$

Problem 5. Find all points on the surface

$$2xy + z^2 = 10$$

where the tangent plane is parallel to the plane $3x + y - 2z = 5$.

$F(x,y,z) = 2xy + z^2$

$\nabla F = \langle 2y, 2x, 2z \rangle$

We are looking for (x,y,z) such that

$\nabla F = \lambda \cdot \langle 3, 1, -2 \rangle$

$$\begin{cases} 2y = 3\lambda \\ 2x = \lambda \\ 2z = -2\lambda \\ 2xy + z^2 = 10 \end{cases} \Longrightarrow \begin{aligned} x &= \tfrac{1}{2}\lambda \\ y &= \tfrac{3}{2}\lambda \\ z &= -\lambda \end{aligned}$$

$2 \cdot \frac{1}{2}\lambda \cdot \frac{3}{2}\lambda + (-\lambda)^2 = 10$

$\frac{5}{2}\lambda^2 = 10, \quad \lambda^2 = 4, \quad \lambda = \pm 2$

If $\lambda = 2$: $(x,y,z) = \boxed{(1, 3, -2)}$

If $\lambda = -2$: $(x,y,z) = \boxed{(-1, -3, 2)}$

Problem 6. a) Find and classify all critical points of the function

$$F(x,y) = x^2 + y^2(1-x)^3$$

$$\nabla F = \langle 2x - 3y^2(1-x)^2, 2y(1-x)^3 \rangle$$

$$\begin{cases} 2x - 3y^2(1-x)^2 = 0 \\ 2y\,(1-x)^3 = 0 \end{cases} \longrightarrow \begin{cases} y = 0 \longrightarrow x = 0 \quad (0,0) \\ x = 1 \longrightarrow x = 0 \quad \text{impossible.} \end{cases}$$

from 1st equation

So $(0,0)$ is the only critical point

$$F_{xx} = 2 + 3y^2 \cdot 2(1-x) \overset{(0,0)}{=} 2$$

$$F_{xy} = F_{yx} = -6y(1-x)^2 \overset{(0,0)}{=} 0$$

$$F_{yy} = 2(1-x)^3 \overset{(0,0)}{=} 2$$

$$\det \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0; \ 2 > 0$$

So $\boxed{(0,0) \text{ is local minimum}}$

b) Find the limit (possibly infinite)

$$\lim_{x\to+\infty} F(x,0) = \lim_{x\to+\infty} x^2 = \boxed{+\infty}$$

b) Find the limit (possibly infinite)

$$\lim_{x\to+\infty} F(x,1) = \lim_{x\to+\infty} \left(x^2 + (1-x)^3\right) = \boxed{-\infty}$$

$$\parallel$$

$$-x^3 + 4x^2 - 3x + 1$$

(Note: F has no global minimum, despite having only one critical point with a local minimum...)

Problem 7. Find and **simplify** F_{xy} for

$$F(x,y,z) = \frac{x-y}{xy - xz - yz + z^2}$$

A "brute force" solution exists. Here is a short-cut:

$$F(x,y,z) = \frac{x-y}{(x-z)(y-z)} = \frac{(x-z)-(y-z)}{(x-z)(y-z)} = \frac{1}{(y-z)} - \frac{1}{(x-z)}$$

$$F_{xy} = \left(\frac{1}{y-z}\right)_{xy} - \left(\frac{1}{x-z}\right)_{xy} = 0_y - \left(\frac{1}{x-z}\right)_{yx} = 0-0 = \boxed{0}$$

Answer the questions in the spaces provided on the question sheets. Explain your answers and reasoning

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Section: ______________________________

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Question:	1	2	3	4	5	6	7	Total
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Score:								

1. (10 points) Find the limit, or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Approaching $(0,0)$ through $y = mx$, $m \neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2 \cdot mx}{x^4+(mx)^2} = \lim_{x\to0} \frac{mx}{x^2+m^2} = 0$$

for any $m \neq 0$.

Approaching $(0,0)$ through $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2 \cdot x^2}{x^4+(x^2)^2} = \frac{1}{2}$$

Therefore, limit does NOT exist.

2. (10 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $yz + x\ln y = z^2$

Let $F(x,y,z) = yz + x\ln y - z^2$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\ln y}{y-2z} = \frac{\ln y}{2z-y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z+\frac{x}{y}}{y-2z} = \frac{z+\frac{x}{y}}{2z-y}$$

3. (15 points) Let $f(x, y) = xe^{xy}$, Find:

a) the value of f_x and f_y at $P = (1, 0)$

b) the tangent plane of $z = f(x, y)$ passes $Q = (1, 0, 1)$

a) $f_x(x,y) = e^{xy} + xye^{xy}$

$f_y(x,y) = x e^{xy} \cdot x = x^2 e^{xy}$

$\Rightarrow f_x(1,0) = 1$

$f_y(1,0) = 1$

b) equation of tangent plane:

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$\Rightarrow z - 1 = 1 \cdot (x - 1) + 1 \cdot (y - 0)$

$\Rightarrow z - 1 = x - 1 + y$

$\Rightarrow z = x + y$

4. (15 points) For function $f(x, y, z) = x^2y + y^2z$ at $P = (1, 2, 3)$

a) Find the directional derivative of f at P in direction $\mu = (2, -1, 2)$.

b) Find the maximum rate of change of f at P and the direction in which it occurs.

a) $\nabla f(x,y,z) = < f_x , f_y , f_z >$

$= < 2xy , x^2 + 2yz , y^2 >$

$\nabla f(1,2 3) = < 4 , 13 , 4 >$

find the unit vector with same direction as $\vec{u} = <2,-1,2>$

$\Rightarrow \vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{<2,-1,2>}{\sqrt{2^2+(-1)^2+2^2}} = < \frac{2}{3} , -\frac{1}{3} , \frac{2}{3} >$

$\Rightarrow D_u f(1,2,3) = \nabla f(1,2,3) \cdot \vec{v}$

$= <4, 13, 4> \cdot < \frac{2}{3} , -\frac{1}{3} , \frac{2}{3} >$

$= 1$

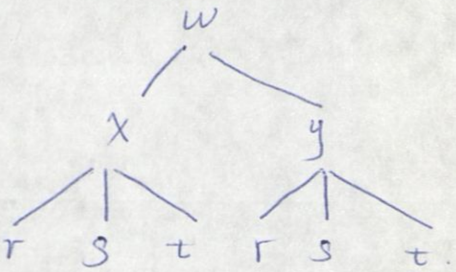
(b) maximum rate of change of f at P is $|\nabla f(1,2,3)|$

$= \sqrt{4^2+13^2+4^2} = \sqrt{201}$

the direction is same as the direction of gradient vector.

$<4, 13, 4>$.

5. (15 points) Let $w = x^3y + y^2$, where $x = rse^t$ and $y = rs^2\cos(t)$. Using the chain rule to find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ at $r = 1, s = 2, t = 0$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$= 3x^2y \cdot se^t + (x^3+2y)\cdot s^2\cos(t)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 3x^2y \cdot re^t + (x^3+2y)\cdot(2rs\cos(t))$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 3x^2y \cdot rse^t + (x^3+2y)\cdot(-rs^2\sin(t))$$

at $r=1, s=2, t=0 \Rightarrow x = 2, y = 4$

$$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{(1,2,0)} = 3\times 2^2\times 4\times 2\times 1 + (2^3+2\times 4)\times 2^2\times 1 = 160$$

$$\left.\frac{\partial w}{\partial s}\right|_{(1,2,0)} = 3\times 2^2\times 4\times 1\times 1 + (2^3+2\times 4)\times(2\times 1\times 2\times 1) = 112$$

$$\left.\frac{\partial w}{\partial t}\right|_{(1,2,0)} = 3\times 2^2\times 4\times 1\times 2\times 1 - (2^3+2\times 4)\times 1\times 2^2\cdot 0$$

$$= 96$$

6. (20 points) For function $f(x, y) = x^3 - 3x + 3xy^2$,

a) Find the linearization of f at (1, 1).

b) Find and classify (as local maximum, local minimum or saddle point) all critical points of f.

a) $L(1,1) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$.

$f(1,1) = 1^3 - 3 \times 1 + 3 \times 1 \times 1^2 = 1$

$f_x(x,y) = 3x^2 - 3 + 3y^2$ $\quad\Rightarrow\quad$ $f_x(1,1) = 3$

$f_y(x,y) = 6xy$ $\qquad f_y(1,1) = 6$

$\Rightarrow L(1,1) = 1 + 3(x-1) + 6(y-1)$

$= 3x + 6y - 8.$

b) ~~f~~ Find critical point:

$f_x(x,y) = 3x^2 - 3 + 3y^2 = 0$ ①

$f_y(x,y) = 6xy = 0$ ②

By ② $x = 0$ or $y = 0$.

If $x = 0$, by ① $y = \pm 1$.

If $y = 0$ by ① $x = \pm 1$

$\Rightarrow$ critial point : $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$

classify critical point : $f_{xx}(x,y) = 6x$ $\quad f_{yy}(x,y) = 6x$. $f_{xy}(x,y) = 6y$

at $(0,1)$. $D = f_{xx}(0,1) f_{yy}(0,1) - f_{xy}(0,1)^2 = 0 - 6^2 = -36 < 0$

$\Rightarrow (0,1)$ is saddle point

at $(0,-1)$ $\quad D = 0 - (-6)^2 = -36 < 0 \Rightarrow (0,-1)$ is saddle point

at $(1,0)$ $\quad D = 36 - 0 > 0$ and $f_{xx}(1,0) > 0 \Rightarrow (1,0)$ is local min

at $(-1,0)$ $\quad D = 36 - 0 > 0$ and $f_{xx}(-1,0) < 0 \Rightarrow (-1,0)$ is local max

7. (15 points) Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2 + y$ on the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

(1) Find the critical point : $f_x(x,y) = 2x = 0$

$f_y(x,y) = 1-2y = 0$

$\Rightarrow (0, \frac{1}{2})$ is a critical point.

and $f(0, \frac{1}{2}) = 0 - (\frac{1}{2})^2 + \frac{1}{2} = \frac{1}{4}$

(2) Find the extreme value on the boundary of D :

It's sufficient to find extreme value of f subject to the constraint $x^2+y^2=1$. Using Lagrange method :

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x & ① \\ 1-2y = \lambda 2y & ② \\ x^2+y^2 = 1 & ③ \end{cases}$$

By ① $x=0$ or $\lambda=1$.

If $x=0$ By ③ $\Rightarrow y = \pm 1$.

If $\lambda = 1$ By ② $\Rightarrow y = \frac{1}{4}$ By ③ $\Rightarrow x = \pm\frac{\sqrt{15}}{4}$

$\Rightarrow (0,1), (0,-1), (\frac{\sqrt{15}}{4}, \frac{1}{4})$ and $(-\frac{\sqrt{15}}{4}, \frac{1}{4})$ satisfy ①②③

$\Rightarrow f(0,1) = 0 \quad f(0,-1) = -2,$

$f(\frac{\sqrt{15}}{4}, \frac{1}{4}) = \frac{9}{8} \quad f(-\frac{\sqrt{15}}{4}, \frac{1}{4}) = \frac{9}{8}$

(3) By (1) and (2)

the absolute max of f is $\frac{8}{9}$ at $(\frac{\sqrt{15}}{4}, \frac{1}{4})$ and $(-\frac{\sqrt{15}}{4}, \frac{1}{4})$

the absolute min of f is -2 at $(0,-1)$.