Problem 4. Let A be the set of all positive integers which have 2025 digits in their decimal representation and all these digits are non-zero. For $a \in A$ let $t(a) = a - m(a)$, where $m(a)$ is the product of all the digits of a. Find a for which $t(a)$ is largest possible.

Solution. Let a be an integer in A for which the value $t(a)$ is largest possible (it exists, since A is a finite set). Let $d_0, d_1, \ldots, d_{2024}$ be the decimal digits of a so $a = d_{2024}10^{2024} + d_{2023}10^{2023} + \ldots + d_1 10 + d_0$. If b is an integer whose decimal digits form a permutation of the digits of a then $m(b) = m(a)$ and $t(a) \geq t(b)$. It follows that $a \geq b$ and, consequently, that $d_{2024} \geq d_{2023} \geq \ldots \geq d_0$. If $d_{2024} < 9$ then $t(a) < a < 9 \cdot 10^{2024} < t(9911\ldots1)$, which contradicts the definition of a. Thus we must have $d_{2024} = 9$.

Suppose that $k < 2024$ and b is obtained from a by replacing the k-th digit d_k of a by another digit $d > 0$. Then $b = a - d_k 10^k + d10^k$ and $m(b) = dm(a)/d_k$, so

$$t(b) = t(a) + (d - d_k)\left(10^k - \frac{m(a)}{d_k}\right)$$

Note that $m(a)/d_k$ is divisible by $9 = d_{2024}$, so $10^k \neq m(a)/d_k$. If $m(a)/d_k < 10^k$, then we must have $d_k = 9$, as otherwise we could take $d = 9$ to get $t(b) > t(a)$. If $m(a)/d_k > 10^k$, then we must have $d_k = 1$, as otherwise we could take $d = 1$ to get $t(b) > t(a)$. In other words, there is an $0 < s < 2024$ such that $d_0 = d_1 = \ldots = d_s = 1$ and $d_{s+1} = \ldots = d_{2024} = 9$ and $10^s < m(a)/d_s = 9^{2024-s}$ and $10^{s+1} > m(a)/d_{s+1} = 9^{2023-s}$. In particular,

$$9^{2024} > 90^s > \frac{9^{2023}}{10} = \frac{9^{2024}}{90}.$$

Taking $log_{90}x = \ln x / \ln 90$ of both sides we get

$$2024\frac{\ln 9}{\ln 90} > s > 2024\frac{\ln 9}{\ln 90} - 1$$

i.e. $s = \left\lfloor 2024\frac{\ln 9}{\ln 90} \right\rfloor = 988$. It follows that $a = 99\ldots911\ldots1$, where we have 1036 9's followed by 989 1's.

Problem. Suppose that we take a subset D of $\{1, 2, \ldots, 9\}$ and consider the set A_D of all 2025-digit integers with all digits in D. Find $a \in A_D$ for which $t(a)$ is largest possible.