Problem 3. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be an even continuous function such that $f(x+2)=f(x)$ for all x and f is increasing on $[0,1]$. Define a new function $g:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$g(x)=\int_0^2 f(t)f(t+x)\mathrm{d}t.$$

Prove that $g(1)$ is the smallest value of g.

Solution. Let us start by recalling the following simple fact: if an integrable function h is periodic with period T then the integral $\displaystyle\int_a^{T+a} h(t)\mathrm{d}t$ does not depend on the value of a. Indeed,

$$\int_a^{T+a} h(t)\mathrm{d}t=\int_a^0 h(t)\mathrm{d}t+\int_0^T h(t)\mathrm{d}t+\int_T^{T+a} h(t)\mathrm{d}t=\int_a^0 h(t)\mathrm{d}t+\int_0^T h(t)\mathrm{d}t+\int_0^a h(s+T)\mathrm{d}s$$

$$=\int_0^T h(t)\mathrm{d}t+\int_a^0 h(t)\mathrm{d}t+\int_0^a h(s)\mathrm{d}s=\int_0^T h(t)\mathrm{d}t$$

(we used the substitution $t=s+T$). Since f is periodic with period 2 and even, we have

$$f(-x)=f(x)\ \text{ and }\ f(1+x)=f(1+x-2)=f(x-1)=f(1-x)$$

for all x. In particular, f is decreasing on the interval $[1,2]$.

First solution (this is a simplified version of the solution by Matt Wolak). Let $A=g(1)-g(x)$. Our goal is to show that $A\le 0$. We have

$$A=\int_0^2 f(t)\left(f(t+1)-f(t+x)\right)\mathrm{d}t.$$

We use the substitution $u=1-t-x$ in the last integral to get

$$A=\int_{1-x}^{-1-x} f(1-u-x)\left(f(2-u-x)-f(1-u)\right)(-\mathrm{d}u)=\int_{-1-x}^{1-x} f(1-t-x)\left(f(2-t-x)-f(1-t)\right)\mathrm{d}t.$$

Since $f(1-t-x)=f(1+t+x)$, $f(2-t-x)=f(-t-x)=f(t+x)$, and $f(1-t)=f(1+t)$, we get

$$A=\int_{-1-x}^{1-x} f(1+t+x)\left(f(t+x)-f(t+1)\right)\mathrm{d}t=-\int_0^2 f(1+t+x)\left(f(t+1)-f(t+x)\right)\mathrm{d}t$$

(since the function $f(1+t+x)(f(1+t)-f(x+t))$ has period 2). Thus

$$2A=\int_0^2 f(t)\left(f(t+1)-f(t+x)\right)\mathrm{d}t-\int_0^2 f(1+t+x)\left(f(t+1)-f(t+x)\right)\mathrm{d}t$$

$$=\int_0^2 \left(f(t)-f(1+t+x)\right)\left(f(t+1)-f(t+x)\right)\mathrm{d}t.$$

We will show that $(f(a)-f(b+1))(f(a+1)-f(b))\le 0$ for any real numbers a,b. This implies that in the last integral we are integrating a non-positive function, hence $2A\le 0$, i.e. $g(1)\le g(x)$. To justify our claim note that any real number can be expressed in the form $m\pm u$ where m is an odd integer and $0\le u\le 1$. In particular, $a=(2k+1)\pm u$ and $b=(2l+1)\pm w$ for some integers k,l and $u,w\in[0,1]$. We have

$$f(a)=f(1\pm u)=f(1-u),\quad f(1+a)=f(\pm u)=f(u),\quad f(b)=f(1-w),\quad f(1+b)=f(w).$$

Since $u,w,1-u,1-w$ are in $[0,1]$ and f is increasing on $[0,1]$ we have

$$f(a)\ge f(b+1)\text{ iff }1-u\ge w\text{ iff }1-w\ge u\text{ iff} f(b)\ge f(1+a).$$

This clearly implies that $(f(a)-f(b+1))(f(a+1)-f(b))\le 0$.

Remark. Note that continuity of f did not play any role in the above solution.

Second solution. We will use the fact that the function g is continuous (since f is continuous). We will discuss this fact at the end of this note.
In order to prove that $g(x) \geq g(1)$ for all x it suffices to show that $g(l/m) \geq g(1)$ for every positive integer m and every integer l (i.e. for every rational number x) by continuity of g. Given l and m let n be a positive integer divisible by m, so $l/m = k/n$ for some integer k ($k = ln/m$). Let

$$S_n = \sum_{i=0}^{2n-1} f\left(\frac{i}{n}\right) f\left(1+\frac{i}{n}\right) \quad \text{and} \quad T_n = \sum_{i=0}^{2n-1} f\left(\frac{i}{n}\right) f\left(\frac{k}{n}+\frac{i}{n}\right).$$

Using the definition of the integral as the limit of Riemann sums, we have

$$g(1) = \lim_{n\to\infty} \frac{S_n}{n} \quad \text{and} \quad g(l/m) = \lim_{n\to\infty} \frac{T_n}{n}$$

(where the second limit is over multiples n of m; the Riemann sums are for the division of the interval $[0,2]$ into $2n$ subintervals of length $1/n$ and the choice of the left end of each interval as the points where our functions are evaluated). It suffices then to show that $S_n \leq T_n$ for all n and all k.
Let $a_{2i+1} = f(i/n)$ for $i = 0, \ldots, n-1$ and $a_{2i} = f(2-i/n) = f(i/n) = a_{2i-1}$ for $i = 1, \ldots, n$. Thus $a_1 \leq a_2 = a_3 \leq a_4 = a_5 \leq \ldots \leq a_{2n-2} = a_{2n-1} \leq a_{2n}$ (we use the fact that f is increasing on $[0,1]$). We have $f(1+i/n) = f(1-i/n) = a_{2(n-i)} = a_{2n+1-(2i+1)}$ for $i = 0, 1, \ldots, n-1$ and $f(1+2-i/n) = f((n-i)/n) = a_{2n+1-2i}$ for $i = 1, \ldots, n$. It follows that

$$S_n = \sum_{i=1}^{2n} a_i a_{2n+1-i}.$$

Since f has period 2, the sequence $f(k/n + i/n)$, $i = 0, 1, \ldots, 2n-1$ is a permutation of the sequence $a_1, a_2, \ldots, a_{2n}$. Thus

$$T_n = \sum_{i=1}^{2n} a_i a_{\sigma(i)}$$

for some permutation σ of the set $1, 2, \ldots, 2n$. The following result, known as the **rearrangement inequality**, implies now that $S_n \leq T_n$:

Rearrangement Inequality. Let $x_1 \leq x_2 \leq \ldots \leq x_m$ and $y_1 \leq y_2 \leq \ldots \leq y_m$ be real numbers. Then

$$x_1 y_m + x_2 y_{m-1} + \ldots + x_m y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \ldots + x_m y_{\sigma(m)} \leq x_1 y_1 + x_2 y_2 + \ldots + x_m y_m$$

for any permutation σ of the set $1, \ldots, m$.

Remark. The solutions submitted by Ashton Keith and Josiah Moltz are both similar to our second solution.

Third solution. Suppose that f is not only continuous, but it has continuous derivative. Then the function g is differentiable and

$$g'(x) = \int_0^2 f(t) f'(t+x) \mathrm{d}t = \int_0^2 f(t-x) f'(t) \mathrm{d}t$$

(we will discuss this fact at the end of this note). Furthermore, since $f(1-x) = f(1+x)$, we have $f'(1+x) = -f'(1-x)$. Now

$$g'(x) = \int_0^2 f(t-x)f'(t)\mathrm{d}t = \int_0^1 f(t-x)f'(t)\mathrm{d}t + \int_1^2 f(t-x)f'(t)\mathrm{d}t = \int_0^1 f(t-x)f'(t)\mathrm{d}t + \int_0^1 f(1+t-x)f'(1+t)\mathrm{d}t$$

$$= \int_0^1 f(t-x)f'(t)\mathrm{d}t - \int_0^1 f(1+t-x)f'(1-t)\mathrm{d}t = \int_0^1 f(t-x)f'(t)\mathrm{d}t - \int_0^1 f(2-u-x)f'(u)\mathrm{d}u$$

$$= \int_0^1 f(t-x)f'(t)\mathrm{d}t - \int_0^1 f(t+x)f'(t)\mathrm{d}t = \int_0^1 (f(t-x) - f(t+x))f'(t)\mathrm{d}t$$

(we used the substitution $u = 1-t$ and the fact that $f(2-t-x) = f(t+x)$). We know that $f'(x) \geq 0$ on $[0,1]$ (since f is increasing). When $x \in [0,1]$ and $t \in [0,1]$ then $f(t-x) \leq f(x+t)$. In fact, if $t+x \leq 1$

then $|x-t| \le t+x \le 1$ and $f(t-x) = f(|t-x|) \le f(t+x)$. If $2 \ge x+t > 1$ than $1 \ge 2-(x+t) \ge |x-t|$ so $f(x+t) = f(2-(x+t)) \ge f(|t-x|) = f(t-x)$. Similarly, when $x \in [1,2]$ and $t \in [0,1]$ then $f(t-x) \ge f(t+x)$. In fact, we can write $x = 2-y$ for some $y \in [0,1]$ and using the previous observation we have $f(t-x) = f(t+y-2) = f(t+y) \ge f(t-y) = f(t+2-y) = f(t+x)$. We see now that $g'(x) \le 0$ when $x \in [0,1]$ and $g'(x) \ge 0$ when $x \in [1,2]$. It follows that g is non-increasing on $[0,1]$ and non-decreasing on $[1,2]$. Thus $g(1)$ is the smallest value of g (since g is periodic with period 2).

To solve the problem for all continuous functions f it suffices to note that any such f can be approximated by continuously differentiable functions which satisfy the conditions of our problem (i.e. are even, periodic with period 2 and increasing on $[0,1]$). We leave the details as an exercise.

Remark. In our second solution, we needed the fact that g is continuous. This can be proved easily using the fact that a continuous function on a closed interval is absolutely continuous. In our third solution we needed the fact that g is differentiable and that $g'(x) = \int_0^2 f(t)f'(t+x)\mathrm{d}t$. This is a bit harder to prove. Both results follow easily from the following general theorem.

Dominated Convergence Theorem. Let (f_n) be a sequence of continuous (or Riemann integrable) functions on $[a,b]$. Suppose there is a number M such that $|f_n(x)| \le M$ for all $x \in [a,b]$ and all n. Suppose furthermore that $f(x)$ is a continuous (or Riemann integrable) function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in [a,b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t.$$

To prove this result within the theory of Riemann integral is not that easy. For a complete proof we recommend the following paper (clickable link):

Three Classical Theorems

(*Three Classical Theorems on Interchanging Limits With Integrals in Calculus* by Sudhir Murthy and Estela Gavosto; https://escholarship.org/content/qt7s93g9d0/qt7s93g9d0.pdf?t=s1wyse)

On the other hand, those familiar with the theory of Lebesque's integral should know that this result is one of the basic theorems of the theory (in fact, one motivation for the development of Lebesque's integral was to make results like the Dominated Convergence Theorem "easy").

Problem 1. Use the Dominated Convergence Theorem to justify the facts used in our second and third solutions.

Problem 2. What is the largest value of g?

Problem 3. Let f_1 and f_2 be two functions as in the problem. Define $g(x)$ by

$$g(x) = \int_0^2 f_1(t)f_2(t+x)\mathrm{d}t.$$

Prove that $g(1)$ is the smallest value of g. What is the largest value of g?

Problem 4. Prove the Rearrangement Inequality (induction on m is a good strategy).